A Generalized Morphological Skeleton Transform Using both Internal and External Skeleton Points

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Abstract - The morphological skeleton transform (MST) is a leading morphological shape representation scheme. In the MST, a given shape is represented as a union of all maximal disks contained in the shape. The concepts of external skeleton points and external maximal disks were introduced recently to derive so-called external shape components for shape matching purposes. In this paper, we develop a generalized morphological skeleton transform that combines the concepts of internal and external maximal disks into a unified framework. In this framework, a shape is described in terms disk components that need to be added as well as disk components that need to be removed. This framework provides a more natural way of modeling the approximation and reconstruction of binary shapes.

Keywords: mathematical morphology, shape representation, skeleton transform, shape approximation, shape reconstruction

1 Introduction

Shape representation is an important issue in image analysis and computer vision. Efficient shape representation provides the foundation for the development of efficient algorithms for many shape-related processing tasks such as image coding, shape matching and recognition, content-based video processing, and image data retrieval.

Mathematical morphology is a shape-based approach to image processing. A number of morphological shape representation algorithms have been proposed [1-8]. The morphological skeleton transform (MST) is a leading morphological shape representation scheme [1]. In the MST, a given shape is represented as a union of all maximal disks contained in the shape. The advantages of the MST include that it has a simple and intuitive mathematical characterization as well as easy and efficient implementations. Some shape matching algorithms have been developed based on the MST [9, 10].

In a recent paper, we developed a structural shape matching algorithm that uses both internal and external shape components [11]. The internal shape components are selected from the internal maximal disks determined by a traditional MST. The external shape components are selected from the external maximal disks determined by a separate “external” skeleton transform.

In this paper, we develop a generalized morphological skeleton transform that combines the concepts of internal and external maximal disks into a unified framework. In this framework, a shape is described in terms disk components that need to be added as well as disk components that need to be removed. The positive (addition) and negative (removal) steps are applied alternately to derive the final representation of the input shape.

2 Internal and External Skeleton Transforms

We first review the standard skeleton transform. For a shape image X and a structuring element B, which is used as the unit disk, if we define size-i disk iB as iB = B ⊕ B ⊕ … ⊕ B (i B’s), then X can be expressed as the union of all maximal disks contained in X:

$$X = (S_N \oplus NB) \cup (S_{N-1} \oplus (N-1)B) \cup \ldots \cup (S_2 \oplus 2B)$$

$$\cup (S_1 \oplus B) \cup S_0$$

where

$$S_i = (X \ominus iB) \setminus ((X \ominus iB) \circ B)$$

and N is the largest integer such that $$X \ominus NB \neq \emptyset$$. Each $$S_i$$ is called a skeleton subset of order i. Each skeleton point in $$S_i$$ represents a maximal disk of size i contained in X. A smoothed (or approximate) version of X can be reconstructed if some lower order skeleton subsets are omitted:

$$X \circ kB = (S_N \oplus NB) \cup (S_{N-1} \oplus (N-1)B) \cup \ldots$$

$$\cup (S_{k+1} \oplus (k+1)B) \cup (S_k \oplus kB), \text{ for } k \leq N.$$  (3)

We can also write

$$X \circ kB = (X \ominus nB) \cup (S_{n+1} \oplus (n-1)B) \cup \ldots \cup$$

$$(S_{k+1} \oplus (k+1)B) \cup (S_k \oplus kB), \text{ for } k < n \leq N. \quad (4)$$

A more accurate approximation $$X \circ kB$$ for X can be obtained from a rougher approximation $$X \circ nB$$ by adding additional maximal disks.

We now review the concept of external skeleton points. For the unit disk B, the reflection of B is defined as

$$B^R = \{b^r | b \in B\}. \quad (5)$$

For a shape image X and the unit disk B, since closing operation is extensive, we have
Fig. 1. Three simple shapes.

\[ X = (X \cdot B^k) \setminus T_0 = ((X \oplus B^k) \ominus B^k) \setminus T_0 \]
\[ = (X_1 \ominus B^k) \setminus T_0 \]
where
\[ T_0 = (X \cdot B^k) \setminus X \]
\[ X_1 = X \oplus B^k. \]

Now we can also write
\[ X_1 = (X_1 \cdot B^k) \setminus T_1 \]
where
\[ T_1 = (X_1 \cdot B^k) \setminus X_1 = ((X \oplus B^k) \ominus B^k) \setminus (X \oplus B^k). \]

Combining (6) with (9), we have
\[ X = (X \cdot mB^k) \setminus (T_{m-1} \oplus (m-1)B) \setminus (T_{m-2} \oplus (m-2)B) \setminus \ldots \]
\[ \setminus (T_1 \oplus B) \setminus T_0 \]
(12)
where
\[ T_i = ((X \oplus IB^k) \ominus B^k) \setminus (X \oplus IB^k). \]

The points in \( T_{m-1}, T_{m-2}, \ldots, T_1, T_0 \) can be viewed as external skeleton points and they represent external disks of different sizes. Removal of such disks results in the restoration of \( X \) from \( X \cdot mB^k \). In fact, these external disks are maximal disks contained in the background shape, or the complement of \( X \). This transform can be viewed as an external skeleton transform. A partially restored \( X \) can be created if some lower order skeleton subsets are omitted:
\[ X \cdot kB^k = (X \cdot mB^k) \setminus (T_{m-1} \oplus (m-1)B) \setminus \ldots \]
\[ \setminus (T_{k+1} \oplus (k+1)B) \setminus (T_k \ominus kB), \text{ for } k < m. \]  

A better approximation \( X \cdot kB^k \) for \( X \) is obtained from a rougher approximation \( X \cdot mB^k \) by removing external maximal disks.

### 3 Generalized Skeleton Transform

It is clear that a shape cannot be completely specified using external skeleton points only. The closed version \( X \cdot mB^k \) of \( X \) in (12) in general will grow bigger as \( m \) increases.

We can use a number of internal skeleton points to represent \( X \cdot mB^k \). Any finite shape can be completely specified using internal skeleton points only. However, sometimes it is more efficient to describe a shape using both internal and external skeleton points. Consider the shape in Fig. 1(a). This is a near circular shape (in digital sense). Therefore, there is an efficient representation for it using internal skeleton points only. Now look at the shape in Fig. 1(b). This is the same circular shape with a near circular hole in it. If we still only use internal skeleton points, then we will need a lot more skeleton points to describe the areas between internal and external boundaries. If we can use external skeleton points to describe the hole first, then we can still use the initial efficient internal skeleton representation for the overall circular shape. Now we consider the shape in Fig. 1(c). The hole on the object has been cut into two halves. To represent two separate holes, we will need to use more external skeleton points. But if we can describe the small line segment that separates the original hole using some positive skeleton points first, then the more efficient representation for the original hole can still be used as part of the overall description. In this section, we develop a generalized skeleton transform that takes this hierarchical and alternately positive and negative description approach.

For a given shape image \( X \) and an structuring element \( B \) which is used as the unit disk, we can write
\[ X = (X \ominus B) \cup S_0 = X' \cup S_0 \] 
where
\[ X' = X \ominus B, \]
\[ S_0 = X \setminus (X \ominus B). \]

In (15), \( X' \) can be viewed as a smoothed version of \( X \) and \( S_0 \) is a positive skeleton subset. For \( X' \) we can write
\[ X' = (X' \cdot B^k) \setminus T_0 = X_1 \setminus T_0 \]
(18)
where
\[ X_1 = X' \cdot B^k \]
\[ T_0 = (X' \cdot B^k) \setminus X' \]
(20)
In (18), \( X_1 \) can be viewed as a smoothed version of \( X' \) and \( T_0 \) is a negative skeleton subset. Combining (15) and (18), we get
Fig. 2. An example showing that \( X_1 \neq X_1 \circ 2B \cup S_1 \oplus B \): (a) Image \( X \); (b) Unit disk \( B \); (c) \( X' = X \circ B \); (d) \( X_1 = X' \bullet B^R \); (e) \( X_1 \circ 2B \cup S_1 \oplus B \).

\[
X = X_1 \setminus T_0 \cup S_0 \tag{21}
\]

We assume that we perform set-theoretical operations from left to right. \( X_1 \) can also be viewed as a smoothed version of \( X \) and both positive and negative skeleton subsets are used to represent \( X \) now. For \( X_1 \) in (21), we have

\[
X_1 \simeq X_1 \circ 2B \cup S_1 \oplus B = X_1' \cup S_1 \oplus B \tag{22}
\]

where

\[
X_1' = X_1 \circ 2B \tag{23}
\]

\[
S_1 = X_1 \oplus B \setminus X_1 \oplus B \tag{24}
\]

We assume that we perform morphological operations from left to right and we perform morphological operations before set-theoretical operations. In (22), \( X_1' \) is a smoothed version of \( X_1 \) and \( S_1 \) is a positive skeleton subset. Note that, in general, \( X_1 \neq X_1 \circ 2B \cup S_1 \oplus B = X_1 \circ B \). An example showing this inequality is given in Fig. 2. Combining (21) and (22), we get an approximation for \( X \):

\[
X \simeq X_1' \cup S_1 \oplus B \setminus T_0 \cup S_0 \tag{25}
\]

Now we have an approximation for \( X \) using a smoothed version \( X_1' \) and a number of skeleton subsets. In the next step, we use closing again. For \( X_1' \) in (25), we have

\[
X_1' \simeq X_1' \bullet 2B^R \setminus T_1 \oplus B = X_2 \setminus T_1 \oplus B \tag{26}
\]

where

\[
X_2 = X_1' \bullet 2B^R \tag{27}
\]

\[
T_1 = X_1' \bullet B^R \setminus X_1' \bullet B^R \tag{28}
\]

Notice also that in general we have \( X_1' \neq X_1' \bullet 2B^R \setminus T_1 \oplus B = X_1' \bullet B^R \). In (26), we have a smoothed version \( X_2 \) and a negative skeleton subset \( T_1 \). Combining (25) and (26), we have a new approximation:

\[
X \simeq X_2 \setminus T_2 \oplus B \cup S_1 \oplus B \setminus T_0 \cup S_0 \tag{29}
\]

In order to see a pattern, we develop two more steps. The next approximation step uses an opening operation and a positive skeleton subset:

\[
X_3 \simeq X_2 \circ 3B \cup S_2 \circ 2B = X_3' \circ S_2 \circ 2B \tag{30}
\]

where

\[
X_3' = X_2 \circ 3B \tag{31}
\]

\[
S_2 = X_2 \oplus 2B \setminus X_2 \oplus 2B \oplus B \tag{32}
\]

A new approximation containing the new skeleton subset is

\[
X \simeq X_3' \cup S_2 \circ 2B \setminus T_2 \setminus B \cup S_1 \setminus B \setminus T_0 \cup S_0 \tag{33}
\]

The next approximation step uses a closing operation and a negative skeleton subset:

\[
X_2' \simeq X_2' \bullet 3B^R \setminus T_2 \setminus B = X_3 \setminus T_2 \setminus B \tag{34}
\]

where

\[
X_3 = X_2' \bullet 3B^R \tag{35}
\]

\[
T_2 = X_2' \circ 2B^R \setminus B^R \setminus X_2' \circ 2B^R \tag{36}
\]

A new approximation containing the new skeleton subset is

\[
X \simeq X_3 \setminus T_2 \setminus 2B \cup S_2 \setminus 2B \setminus T_1 \setminus B \cup S_1 \setminus B \setminus T_0 \cup S_0 \tag{37}
\]

In general, with

\[
X \simeq X_1 \setminus T_{i+1} \circ (i-1)B \cup S_{i+1} \circ (i-1)B \setminus \ldots \setminus T_0 \cup S_0 \tag{38}
\]

we use

\[
X_1 \simeq X_1 \circ (i+1)B \cup S_i \oplus iB = X_i' \cup S_i \oplus iB \tag{39}
\]

where

\[
X_i' = X_1 \circ (i+1)B \tag{40}
\]

\[
S_i = X_1 \oplus iB \setminus X_1 \oplus iB \tag{41}
\]

A new approximation for \( X \) containing \( S_i \) is

\[
X \simeq X_i' \cup S_i \oplus iB \setminus T_{i+1} \circ (i-1)B \cup S_{i+1} \circ (i-1)B \setminus \ldots \setminus T_0 \cup S_0 \tag{42}
\]

We also use

\[
X_{i+1} \simeq X_i' \bullet (i+1)B^R \setminus T_i \oplus iB = X_{i+1} \setminus T_i \oplus iB \tag{43}
\]

where

\[
X_{i+1} = X_i' \bullet (i+1)B^R \tag{44}
\]

\[
T_i = X_i' \setminus B^R \setminus B^R \setminus X_i' \setminus B^R \tag{45}
\]

A new approximation including the latest skeleton subset is

\[
X \simeq X_{i+1} \setminus T_i \oplus iB \cup S_i \oplus iB \setminus T_{i+1} \circ (i-1)B \cup S_{i+1} \circ (i-1)B \setminus \ldots \setminus T_0 \cup S_0 \tag{46}
\]

Eventually, we will encounter a \( N \) such that \( X_N' = X_N \circ (N+1)B = O \) with \( X_N \neq O \). This implies that \( X_N \oplus (N+1)B = O \). And from this, we can see that \( S_N = X_N \oplus NB \). It is also easy to see that \( T_N = O \) and \( X_N = X_N' = O \) for \( n \geq N+1 \). So the final approximation is

\[
X \simeq S_N \circ NB \setminus T_{N+1} \circ (N-1)B \cup S_{N+1} \circ (N-1)B \setminus \ldots \setminus T_0 \cup S_0 \tag{47}
\]

We have obtained a series of approximations for \( X \). The final approximation uses a sequence of alternately positive and negative skeleton subsets. These skeleton subsets are obtained from progressively smoothed versions of \( X \). We now claim that all these approximations are in fact exact representations of \( X \):

\[
X = X' \circ S_0
\]

\[
= X_1 \setminus T_0 \cup S_0
\]

\[
= X_i' \cup S_i \oplus B \setminus T_0 \cup S_0
\]

\[
= X_2 \setminus T_1 \oplus B \cup S_1 \oplus B \setminus T_0 \cup S_0
\]

\[
= X_3 \setminus T_2 \setminus B \setminus S_2 \setminus B \setminus T_1 \setminus B \cup S_1 \setminus B \setminus T_0 \cup S_0
\]

\[
= S_N \circ NB \setminus T_{N+1} \circ (N-1)B \cup S_{N+1} \circ (N-1)B \setminus \ldots \setminus T_0 \cup S_0 \tag{48}
\]

where \( X_i' \), \( X_n \), \( S_i \), and \( T_i \) are defined in (16, 17), (19, 20), (40, 41), and (44, 45).

We first show that all the approximations contain \( X \) as a subimage. Following the derivation process, we first have

\[
X = X' \circ B \cup S_0 = X' \cup S_0 = X = X \circ B
\]

We then have

\[
X' = X' \bullet B^R \setminus T_0 = X_1 \setminus T_0 \text{ and } X = X_1 \setminus T_0 \cup S_0 = X_1 = X' \bullet B^R.
\]

Now consider the approximation step \( X_1 = X_1 \circ 2B \cup S_1 \oplus B \)
Fig. 3. The skeleton points of the standard and the new skeleton transforms.

\[ X_1' \cup S_1 \oplus B. \] Note that \( X_1 = X' \bullet B^k \supseteq X' \) and \( X_1 \circ 2B \cup S_1 \oplus B = X_1' \cup S_1 \oplus B = X_1 \circ B \) contains all size-one disks in \( X_1 \). So \( X_1' \cup S_1 \oplus B = X_1 \circ B \) contains all size-one disks in \( X_1 \).

From \( X' = X' \bullet B^k \setminus T_0 \), we see that \( T_0 \) does not contain any points in \( X' \). Thus \( X_1' \cup S_1 \oplus B \setminus T_0 \) still contains all size-one disks in \( X' \). Note also that \( X' = X \circ B \) is a union of size-one disks. Therefore, \( X_1' \cup S_1 \oplus B \setminus T_0 \supseteq X' \).

Combining with \( \approx \) and \( \approx \), we get
\[ X_1' \cup S_1 \oplus B \setminus T_0 \cup S_0 \supseteq X. \]

Consider the next approximation step \( X_1' \approx X_1' \bullet 2B^k \setminus T_1 \oplus B = X_1 \setminus T_1 \oplus B. \) Note that \( X_1' \bullet 2B^k \setminus T_1 \oplus B = X_1 \setminus T_1 \oplus B = X_1' \bullet B^k. \)

Therefore
\[ X_2' \setminus T_1 \oplus B \supseteq X_1'. \]

Combining this with (49), we get \( X_2 \setminus T_1 \oplus B \cup S_1 \oplus B \setminus T_0 \cup S_0 \supseteq X. \) Move on to the next approximation step \( X_2 = X_2' \circ 2B \cup S_2 \oplus 2B = X_2' \oplus S_2 \oplus 2B. \) Note that \( X_2 = X_1' \bullet 2B^k \supseteq X_1' \) and \( X_2 \circ 2B \cup S_2 \oplus 2B = X_2' \cup S_2 \oplus 2B = X_2 \circ 2B \) contains all size-two disks in \( X_2 \). So \( X_2' \cup S_2 \oplus 2B = X_2 \) also contains all size-two disks in \( X_2' \).

From \( X_2 \setminus T_1 \oplus B \supseteq X_1' \) in (50), \( T_1 \oplus B \) does not contain any points in \( X_1' \). Thus \( X_2' \cup S_2 \oplus 2B \setminus T_1 \oplus B \) still contains all size-two disks in \( X_1' \). Note also that \( X_1' = X_1 \circ 2B \) is a union of size-two disks. Therefore \( X_1' \cup S_1 \oplus 2B \setminus T_1 \oplus B \supseteq X_1'. \)

Combining this with (49), we get \( X_2' \cup S_2 \oplus 2B \setminus T_1 \oplus B \cup S_1 \oplus 2B \setminus T_0 \cup S_0 \supseteq X. \) Repeating similar steps, we can show that all our approximations contain \( X \) as a subimage. That means that all the original approximations for \( X \) are subimages of \( X \). Combined with the earlier results that all the original approximations contain \( X \) as a subimage, we conclude that all these approximations are exact representations of \( X \).

4 Representation Examples

We now go back to the shape in Fig. 1(c). The standard skeleton transforms uses 151 internal skeleton points, as shown in Fig. 3(a), to represent this shape. Our new algorithm only uses 20 internal and external skeleton points. Fig. 3(b) shows the internal skeleton points and Fig. 3(c) shows the external skeleton points of the new algorithm. In our implementation of both algorithms, we use two structuring elements \( B_0 \) and \( B_1 \) shown in Fig. 4 to define discrete disks of different sizes:

\[ B = B_0 \]

\[ iB = (i-1)B \oplus B_{(i-1) \text{mod } 2}, \text{ for } i \geq 2 \]

The skeleton subset formulas now become
We have developed a generalized skeleton transform that represents a binary shape using both internal and external skeleton points. Using this representation, a shape can be approximated at different levels. The roughest approximation is obtained by applying opening and closing operations alternately to the original shape using disk structuring elements of increasing sizes. More accurate approximations can be reconstructed by alternately adding and removing disk parts of decreasing sizes from the initial approximation. The standard skeleton transform can be seen as a special case of our algorithm. In the standard skeleton transform, only opening operations are used in deriving skeleton subsets and only “addition” (set union) operations are used to rebuild more accurate approximations.

6 References


5 Conclusions


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**Fig. 5.** A dog shape and its reconstructions.

**Fig. 6.** A fractal shape and its reconstructions.